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Stability and Boundedness in Difference Systems with Finite Delay

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Abstract—In this paper, stability and boundedness theorems for delay difference systems with the condition

$$\Delta V(n, x(n)) \leq -W_2(|x(n)|) + \sigma$$

are given, where Δ is the backward difference operator. Some known results are generalized. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Uniform asymptotic stability, Uniform boundedness, Uniform ultimate boundedness, Difference systems.

1. INTRODUCTION

In the last few years much research has been made on the stability and boundedness of functional differential equations with finite delay. See, for instance, [1–11]. For the general theory of functional differential equations, one is referred to [12–17]. However, the results about the stability and boundedness of difference equations with finite delay are very scarce in the literature (see [18–20]). It is well known, that the properties of differential equations and their discrete analogues can be quite different. For example, every solution of the logistic equation

$$x'(t) = rx(t)\left(1 - \frac{x(t)}{K}\right)$$

is monotonic. But its discrete analogue

$$x(n+1) = ax(n)(1 - x(n))$$

has a chaotic solution when $a = 4$ (see [21]).

In this paper, we consider the difference systems with finite delay of the form

$$x(n+1) = f(n, x_n), \qquad \text{for } n \in Z, \tag{1}$$

where $f : Z \times C_d \to \mathbb{R}^k$ and $f(n, 0) \equiv 0$, for all $n \in Z$ such that equation (1) has the zero solution $x(n) \equiv 0$. (For considering the boundedness of solutions of equation (1), we need not the condition $f(n, 0) \equiv 0$, for all $n \in Z$.) Z is the integer set and C_d is the space of functions $\phi : Z^{|-\ell,0|} \to \mathbb{R}^k$, $Z^{|-\ell,0|} := \{-\ell, \dots, -1, 0\}$, and x_n is defined as $x_n(m) = x(n+m)$, for all $m \in Z^{|-\ell,0|}$, so $x_n \in C_d$ for every $n \in Z$.

For any $M > 0$, let

$$C_d(M) := \{\phi \in C_d, \|\phi\| < M\},$$

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX where $\|\phi\| = \sup\{|\phi(m)|, m \in Z^{|-\ell,0|}\}$ and $|\cdot|$ is any norm in the k-dimensional Euclidean space $\mathbb{R}^k$.

It is obvious, that for any $n_0 \in Z^+$, $Z^+ := \{0, 1, \dots\}$, and any given $\phi \in C_d(M)$, there exists a unique solution of equation (1), denoted by $x(n, n_0, \phi)$ or $x(n)$ in short, such that $x_{n_0} = \phi(m)$ for $(m \in Z^{|-\ell,0|}$ and $x(n)$ satisfies equation (1), for all $n \geq n_0$.

In the following, we give some definitions.

DEFINITION 1. *The zero solution of equation (1) is said to be stable if for every $\varepsilon > 0$ and any $n_0 \in Z^+$, there exists a $\delta > 0$ such that if $\phi \in C_d(\delta)$, then $|x(n)| < \varepsilon$, for all $n \geq n_0$.*

DEFINITION 2. *The zero solution of equation (1) is said to be uniformly stable, if the δ in Definition 1 is independent of n_0.*

DEFINITION 3. *The zero solution of equation (1) is said to be asymptotically stable, if it is stable and there exists an $\eta > 0$ such that for every $\gamma > 0$ and any $n_0 \in Z^+$, there exists an $N \in Z^+ \setminus \{0\}$ such that if $\phi \in C_d(\eta)$ then $|x(n)| < \gamma$, for all $n \geq n_0 + N$.*

DEFINITION 4. *The zero solution of equation (1) is said to be uniformly asymptotically stable, if it is uniformly stable and the N in Definition 3 is independent of n_0.*

DEFINITION 5. *The solutions of equation (1) are said to be uniformly bounded, if for every $B_1 > 0$ there exists a $B_2 > 0$ such that if $\phi \in C_d(B_1)$, then $|x(n)| < B_2$, for all $n \geq n_0$.*

DEFINITION 6. *The solutions of equation (1) are said to be ultimately bounded for bound B, if for every $B_0 > 0$ and any $n_0 \in Z^+$ there exists an $N \in Z^+ \setminus \{0\}$ such that if $\phi \in C_d(B_0)$, then $|x(n)| < B$, for all $n \geq n_0 + N$.*

DEFINITION 7. *The solutions of equations (1) are said to be uniformly ultimately bounded, for bound B if the N in Definition 6 is independent of n_0.*

DEFINITION 8. *A continuous, strictly increasing function $W : \mathbb{R}^+ \to \mathbb{R}^+$ with $W(0) = 0$ and $W(s) > 0$ if $s > 0$ is said to be a wedge function, or wedge in short.*

In the sequel, we define that Δ is the backward difference operator i.e., $\Delta x(n) = x(n) - x(n-1)$. To the knowledge of the authors, there is no paper which uses the backward difference operator instead of the forward difference operator in the study of stability and boundedness.

2. STABILITY

THEOREM 1. *Let W_1, W_2, and W_3 be wedges. Suppose that there exists a $V : Z^+ \times \mathbb{R}^k \to \mathbb{R}^+$ such that*

(i) *$W_1(|x(n)|) \leq V(n, x(n)) \leq W_2(|x(n)|)$ for $n \in Z^+$;*
(ii) *for some $\beta_0 > 0$, any $\alpha > 0 (0 < \alpha \leq \beta_0)$ and any $\sigma > 0$, there exists an $\eta = \eta(\alpha, \beta_0, \sigma) > 0$ such that when $\alpha \leq V(n, x(n))$, $\sup_{m \in \{-\ell,\dots,0\}} V(n+m, x(n+m)) \leq \beta_0$ and $V(n+m, x(n+m)) \leq V(n, x(n)) + \eta$ for $m \in \{-\ell, \dots, -1, 0\}$, we have*

$$\Delta V(n, x(n)) \leq -W_3(|x(n)|) + \sigma.$$

Then, the zero solution of equation (1) is uniformly asymptotically stable.

PROOF. We first prove the uniform stability.

For any $\varepsilon > 0$ which satisfies $W_1(\varepsilon) < \beta_0$, we select a $\delta > 0$ such that $W_2(\delta) < W_1(\varepsilon)$.

For any $n_0 \in Z^+$, $\phi \in C_d(\delta)$, let $x(n) := x(n, n_0, \phi)$ be the solution of equation (1). From (i) we have

$$W_1(|x(n)|) \leq V(n, x(n)) \leq W_2(\delta) < W_1(\varepsilon), \qquad \text{for } n \in \{n_0 - \ell, \dots, n_0\}.$$

We will prove in the following

$$V(n,x(n)) < W_1(\varepsilon), \qquad \text{for all } n \geq n_0. \tag{2}$$

If it is not the case, then there exists an $n_1 > n_0$ such that

$$\begin{aligned} V(n_1,x)(n_1)) \geq W_1(\varepsilon) > W_2(\delta), \qquad &\text{and} \\ V(n,x(n)) \leq V(n_1,x(n_1)), \qquad &\text{for all } n \in \{n_0+1,\ldots,n_1-1\}. \end{aligned}$$

Thus, this follows $|x(n_1)| \geq \delta$ and $\Delta V(n_1,x(n_1)) \geq 0$.

On the other hand, we let $\sigma > 0$ be such that $\sigma < W_3(\delta)$ and $\alpha = W_2(\delta) \leq \beta_0$. Then, we have $\alpha \leq V(n_1,x(n_1))$, $\sup_{m\in\{-\ell,\ldots,0\}} V(n_1+m,x(n_1+m)) \leq \beta_0$, and $V(n_1+m,x(n_1+m)) \leq V(n_1,x(n_1))+\eta$ for any $\eta > 0$ and all $m \in \{n_1-\ell\ldots,n_1\}$. Consequently, we have from (ii)

$$\Delta V(n_1,x(n_1)) \leq \sigma - W_3(|x(n_1)|) < 0.$$

This is a contradiction. Therefore, (2) holds and uniform stability follows.

Let us now prove the uniform asymptotic stability.

Let $\delta > 0$ and $M > 0$ be such that $2W_2(\delta_0) < W_1(M) \leq \beta_0$. We can similarly prove that if $\phi \in C_d(\delta_0)$, then $|x(n)| \leq M$ and $V(n,x(n)) \leq W_2(\delta_0) \leq \beta_0$, for all $n \geq n_0$.

For any $\varepsilon > 0$ ($\varepsilon < M$), we will prove that there exists an $N \in Z^+ \setminus \{0\}$ such that for any $n_0 \in Z^+$ and any given $\phi \in C_d(\delta_0)$ we have

$$V(n,x(n)) < W_1(\varepsilon), \qquad \text{for all } n \geq n_0 + N.$$

Let $\sigma = (W_3(\varepsilon)/2)$, $\alpha = W_1(\varepsilon)$. There will exist an $\eta = \eta(\alpha,\beta_0,\sigma) > 0$. Suppose, that N' is the least positive integer such that $W_1(\varepsilon) + N'\eta > W_2(\delta_0)$. Write

$$n_K = \left[n_0 + K\left(\ell + \frac{W_2(\delta_0)}{\sigma}\right)\right] + 1, \qquad \text{for } K = 0,1,\ldots,N',$$

where $[\cdot]$ is the integer function. We will prove by induction

$$V(n,x(n)) < W_1(\varepsilon) + (N'-K)\eta, \qquad \text{for } n \geq n_K \quad \text{and} \quad K = 0,1,\ldots,N'. \tag{3}$$

It is easy to see that (3) holds for $K = 0$. Suppose that (3) holds for some $K (0 \leq K < N')$. Then, by induction we must show that

$$V(n,x(n)) < W_1(\varepsilon) + (N'-K-1)\eta, \qquad \text{for } n \geq n_{K+1}.$$

As a matter of fact, let $\bar{n}_0 \geq n_K + \ell$ be such that $V(\bar{n}_0,x(\bar{n}_0)) \geq W_1(\varepsilon) + (N'-K-1)$. Then, we have $\alpha = W_1(\varepsilon) \leq V(\bar{n}_0,x(\bar{n}_0))$, $\sup_{m\in\{-\ell,\ldots,0\}} V(\bar{n}_0+m,x(\bar{n}_0+m)) \leq W_2(\delta_0) \leq W_1(M) \leq \beta_0$, and $V(\bar{n}_0+m,x(\bar{n}_0+m)) \leq W_1(\varepsilon) + (N'-K)\eta \leq V(\bar{n}_0,x(\bar{n}_0)) + \eta$ for $m \in \{-\ell,\ldots,-1,0\}$. From (ii) and definition of σ, we have

$$\Delta V(\bar{n}_0,x(\bar{n}_0)) \leq -W_3(|x(\bar{n}_0)|) + \sigma \leq -2\sigma + \sigma = -\sigma < 0.$$

The above proof shows that if for some $\bar{n}_1 \geq n_K + \ell$ we have $V(\bar{n}_1,x(\bar{n}_1)) < W_1(\varepsilon) + (N'-K-1)\eta$, then there must be $V(n,x(n)) < W_1(\varepsilon) + (N'-K-1)\eta$ for $n \geq \bar{n}_1$. In the following, we prove the existence of $\bar{n}_1$. In fact, if there exists an $\bar{n}_2 \in Z^+ \setminus \{0\}$ such that $V(n,x(n)) \geq W_1(\varepsilon) + (N'-K-1)\eta$, for all $n_K + \ell \leq n \leq \bar{n}_2$, then we have by use of similar reference from the above

$$\Delta V(n,x(n)) \leq -W_3(|x(n)|) + \sigma \leq -\sigma < 0, \qquad \text{for } \left[n_{K+1} - \frac{W_2(\delta_0)}{\sigma}\right] \leq n \leq \bar{n}_2.$$

Therefore,

$$V(\bar{n}_2, x(\bar{n}_2)) \le W_2(\delta_0) - \sigma\left(\bar{n}_2 - n_{K+1} + \frac{W_2(\delta_0)}{\sigma}\right) \le \sigma(n_{K+1} - \bar{n}_2),$$

and $\bar{n}_2 \le n_{K+1}$ from the nonnegativity of V, i.e., there exists an $\bar{n}_1 \in \{n_K + \ell, \dots, n_{K+1}\}$ such that $V(\bar{n}_1, x(\bar{n}_1)) < W_1(\varepsilon) + (N' - K - 1)\eta$. So, we have $V(n, x(n)) < W_1(\varepsilon) + (N' - K - 1)\eta$ for $n \ge n_{K+1} \ge \bar{n}_1$. By the principle of induction, (3) holds, for all $K \in \{0, 1, \dots, N'\}$. Especially, we have

$$V(n, x(n)) < W_1(\varepsilon) \qquad \text{for } n \ge n_{N'}.$$

Let $N = [N'(\ell + (W_2(\delta_0)/\sigma))] + 1$. It is easy to see that N is independent of n_0 and ϕ, and

$$W_1(|x(n)|) \le V(n, x(n)) < W_1(\varepsilon), \qquad \text{for all } n \ge n_0 + N, \qquad \text{or}$$

$$|x(n)| < \varepsilon, \qquad \text{for } n \ge n_0 + N.$$

This completes the proof. ∎

COROLLARY 1. *Let W_1, W_2, and W_3 be the same as in Theorem 1. Suppose that there exists a $V : Z^+ \times \mathbb{R}^k \to \mathbb{R}^+$ and a continuous, nondecreasing function $p : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies $p(s) > s$ for $s > 0$ such that*

(i) *$W_1(|x(n)|) \le V(n, x(n)) \le W_2(|x(n)|)$ for $n \in Z^+$,*
(ii) *when $V(n + m, x(n + m)) < p(V(n, x(n)))$ for $m \in \{-\ell, \dots, -1, 0\}$,*

we have

$$\Delta V(n, x(n)) \le -W_3(|x(n)|).$$

Then, the zero solution of equation (1) is uniformly asymptotically stable.

PROOF. We need only to prove that Condition (ii) in the corollary implies Condition (ii) in Theorem 1.

For any $\alpha, \beta (0 < \alpha < \beta)$ and $\sigma > 0$, let $\eta = \inf_{\alpha \le s \le \beta}(p(s) - s)$. It is easy to see that $\eta > 0$. If $\alpha \le V(n, x(n))$, $\sup_{m \in \{-\ell, \dots, 0\}} V(n+m, x(n+m)) \le \beta$, and $V(n+m, x(n+m)) \le V(n, x(n)) + \eta$ for $m \in \{-\ell, \dots, -1, 0\}$, then

$$\begin{aligned} V(n + m, x(n + m)) &\le V(n, x(n)) + \eta \\ &\le V(n, x(n)) + p(V(n, x(n))) - V(n, x(n)) \\ &= p(V(n, x(n))), \qquad \text{for } m \in \{-\ell, \dots, -1, 0\}. \end{aligned}$$

Hence, we have from (ii)

$$\Delta V(n, x(n)) \le -W_3(|x(n)|) + \sigma.$$

The proof is complete. ∎

REMARK 1. Corollary 1 is the discrete analogue of a well-known classical Razumikhin-type theorem (see, for instance, [12–17]). Also, the Δ in Corollary 1 can be the forward difference operator.

COROLLARY 2. *Let W_1, W_2, and W_3 be the same as in Theorem 1. If there exists a $V : Z^+ \times \mathbb{R}^k \to \mathbb{R}^+$ such that*

(i) *$W_1(|x(n)|) \le V(n, x(n)) \le W_2(|x(n)|)$ for $n \in Z^+$;*
(ii) *$\Delta V(n, x(n)) \le G(V(n, x(n)), \sup_{m \in \{-\ell, \dots, 0\}} V(n+m, x(n+m)))$, where $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and $G(V, V) = -W_3(V)$ for $V > 0$, then the zero solution of equation (1) is uniformly asymptotically stable.*

PROOF. We need only to prove that Condition (ii) in the corollary implies Condition (ii) in Theorem 1.

For any $\alpha, \beta (0 < \alpha \leq \beta)$ and $\sigma > 0$, there exists an $\eta = \eta(\alpha, \beta, \sigma) > 0$ such that for any $s, t \in [\alpha, \beta]$ and $|s - t| \leq \eta$, we have $|G(s,t) - G(s,s)| \leq \sigma$ because of the uniform continuity of $G(s,t)$ on $[\alpha, \beta] \times [\alpha, \beta]$. Let $\alpha \leq V(n, x(n))$, $\sup_{m \in \{-\ell, \dots, 0\}} V(n+m, x(n+m)) \leq \beta$, and $\sup_{m \in \{-\ell, \dots, 0\}} V(n+m, x(n+m)) \leq V(n, x(n)) + \eta$. Then, we have $|\sup_{m \in \{-\ell, \dots, 0\}} V(n+m, x(n+m)) - V(n, x(n))| < \eta$. This follows

$$
\begin{aligned}
\Delta V(n, x(n)) \leq & - W_3(V(n, x(n))) \\
& + |G(V(n, x(n)), \sup_{m \in \{-\ell, \dots, 0\}} V(n+m, x(n+m))) \\
& - G(V(n, x(n)), V(n, x(n)))| \leq -W^*(|x(n)|) + \sigma,
\end{aligned}
$$

where $W^*(s) = \inf_{W_1(s) \leq \theta \leq W_2(s)} W_3(\theta)$. So, the zero solution of equation (1) is uniformly asymptotically stable.

REMARK 2. The Δ in Corollary 2 can be the forward difference operator. Furthermore, we have the following theorem.

THEOREM 2. *Suppose that all conditions but (ii) in Theorem 1 hold. And suppose that (ii)* $\Delta V(n, x(n)) \leq -p(n)(W_3(|x(n)|) - \sigma)$. *Then,*

(1) *if* $p : Z^+ \to \mathbb{R}^+$ *and* $\sum_{m=0}^{\infty} p(m) = \infty$, *then the zero solution of equation (1) is uniformly stable and asymptotically stable;*
(2) *if* $p : Z^+ \to \mathbb{R}^+$ *and for any* $L > 0$, *there exists an* $L' \in Z^+ \setminus \{0\}$ *such that* $\sum_{m=n}^{n+L'} p(m) > L$ *for* $n \in Z^+$, *then the zero solution of equation (1) is uniformly asymptotically stable.*

The proof of the above theorem is similar to that of Theorem 1, so we omit it. ∎

COROLLARY 3. *Let* W_1, W_2, *and* W_3 *be the same as in Theorem 1. If there exists a* $V : Z^+ \times \mathbb{R}^k \to \mathbb{R}^+$ *which satisfies*

(i) $W_1(|x(n)|) \leq V(n, x(n)) \leq W_2(|x(n)|)$ *for* $n \in Z^+$,
(ii) $\Delta V(n, x(n)) \leq p(n) G(V(n, x(n)), \sup_{m \in \{-\ell, \dots, 0\}} V(n+m, x(n+m)))$, *where* $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is continuous and* $G(V, V) = -W_3(V)$ *for* $V > 0$,

then

(1) *if* $p : Z^+ \to \mathbb{R}^+$ *and* $\sum_{m=0}^{\infty} p(m) = \infty$, *then the zero solution of equation (1) is uniformly stable and asymptotically stable,*
(2) *if* $p : Z^+ \to \mathbb{R}^+$ *and for any* $L > 0$, *there exists an* $L' \in Z^+ \setminus \{0\}$ *such that* $\sum_{m=n}^{n+L'} p(m) > L$ *for* $n \in Z^+$, *then the zero solution of equation (1) is uniformly asymptotically stable.*

REMARK 3. The Δ in Corollary 3 can also be the forward difference operator.

3. BOUNDEDNESS

THEOREM 3. *let* W_1, W_2, *and* W_3 *be the same as in Theorem 1 and* $W_1(s) \to 0$ *(as* $s \to \infty$*). Suppose that there exists a* $V : Z^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

(i) $W_1(|x(n)|) \leq V(n, x(n)) \leq W_2(|x(n)|)$ *for* $n \in Z^+$,
(ii) *for some* $\alpha_0 > 0$, *any* $\beta \leq \alpha_0$ *and any* $\sigma > 0$, *there exists an* $\eta = \eta(\alpha_0, \beta, \sigma) > 0$ *such that when* $\alpha_0 \leq |x(n)|$, $\|x_n\| \leq \beta$, *and* $V(n+m, x(n+m)) \leq V(n, x(n)) + \eta$, *for all* $m \in \{-\ell, \dots, -1, 0\}$, *we have*

$$\Delta V(n, x(n)) \leq -W_3(|x(n)|) + \sigma.$$

Then, the solutions of equation (1) are uniformly bounded and uniformly ultimately bounded.

PROOF. Let $B_1 \geq \alpha_0$ be given. We select a $B_2 > 0$ such that $W_2(B_1) < W_1(B_2)$. For any $n_0 \in Z^+$ and any $\phi \in C_d(B_1)$, let $x(n) := x(n, n_0, \phi)$. Let $\beta = B_2$, $\alpha_0 = B_1$, and $0 < \sigma < W_3(B_1)$. Then, there exists an $\eta = \eta(\alpha_0, \beta, \sigma) > 0$. So,

$$V(n, x(n)) < W_2(B_1) < W_1(B_2), \quad \text{for all } n \geq n_0.$$

In fact, if there exists an $n_1 > n_0$ such that $V(n, x(n)) \geq W_1(B_2) > W_2(B_1)$ and $V(n, x(n)) < W_1(B_2)$, for all $n < n_1$, then $\Delta V(n_1, x(n_1)) > 0$. It follows that $|x(n-1)| > B_1$.

On the other hand, because $|x(n_1)| \geq B_1$, $\|x_n\| \leq B_2$, and $V(n_1+m, x(n_1+m)) \leq V(n, x(n))+\eta$ for $m \in \{-\ell, \ldots, -1, 0\}$, so we have

$$\Delta V(n_1, x(n_1)) \leq -W_3(|x(n_1)|) - \sigma < 0.$$

This is a contradiction. So, the uniform boundedness follows.

Next, we will prove the uniform ultimate boundedness.

Let $B > 0$ be such that $W_1(B) = W_2(\alpha_0)$. For any $B_3 \geq \alpha_0$, there exists a $B_4 > B$ such that for every $n_0 \in Z^+$, any $\phi \in C_d(B_3)$, we have

$$V(n, x(n)) \leq W_1(B_4) \quad \text{and} \quad |x(n)| \leq B_4, \qquad \text{for all } n \geq n_0.$$

Let $\beta = B_4$, $\sigma = (W_3(\alpha_0)/2)$, and N' be the least positive integer such that $W_1(B) + N'\eta > W_1(B_4)$ for $\eta = \eta(\alpha_0, \beta, \sigma)$.

Define $n_0 = n_0$ and $n_K > n_{K-1} + \ell$ such that

$$n_K - n_{K-1} - \ell = \left\lceil \frac{W_2(B_4)}{\sigma} \right\rceil, \qquad \text{for } K = 1, \ldots, N'.$$

We can prove by following the proof of Theorem 1 that

$$V(n, x(n)) < W_1(B) + (N' - K)\eta, \qquad \text{for } n \geq n_K \quad \text{and} \quad K = 0, 1, \ldots, N'.$$

(If we select $N = n_{N'} - n_0$, then $|x(n)| \leq B$ for $n \geq n_0 + N$, where N may depend on n_0.)

To prove the uniform ultimate boundedness, we let $L = [(W_2(B_4)/\sigma)] + 1$ be given and $L' > L$ be an integer. Now, we define $N_K = n_K - n_0$. It is obvious that $N_0 = 0$ and $N_K - N_{K-1} - \ell = [(W_2(B_4)/\sigma)]$. It follows

$$\begin{aligned} L' &> N_K - N_{K-1} - \ell, && \text{for } K = 1, \ldots, N', \qquad \text{or} \\ N_K &< L' + N_{K-1} = \ell, && \text{for } K = 1, \ldots, N'. \end{aligned}$$

Consequently, we have

$$N_{N'} \leq N_0 + N'(L' + \ell) = N'(L' + \ell).$$

Now, let $N = N'(L' + \ell)$. Then, from above we have

$$|x(n)| < B, \qquad \text{for } n \geq n_0 + N \geq n_0 + N_{N'}.$$

It is easy to see that N is independent of $n_0 \in Z^+$. This completes the proof. ∎

COROLLARY 4. *Let W_1, W_2, and W_3 be the same as in Theorem 1. If there exist a $V : Z^+ \times \mathbb{R}^k \to \mathbb{R}^+$, a continuous, nondecreasing function $p : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies $p(s) > s$ for $s > 0$, and a constant $H \geq 0$ such that*

(i) *$W_1(|x(n)|) \leq V(n, x(n)) \leq W_2(|x(n)|)$ for $n \in Z^+$,*

(ii) *when $|x(n+m)| \geq H$ and $V(n+m, x(n+m)) < p(V(n, x(n)))$ for $m \in \{-\ell, \ldots, -1, 0\}$, we have*

$$\Delta V(n, x(n)) \leq -W_3(|x(n)|),$$

then the solutions of equation (1) are uniformly bounded and uniformly ultimately bounded.

REMARK 4. Corollary 4 is the discrete analogue of another well-known classical Razumikhin-type theorem (see, for instance, [12–17]). The Δ in Corollary 4 can be also the forward difference operator.

We also have the following statement.

THEOREM 4. *Suppose that all conditions but (ii) in Theorem 3 hold. And suppose that*

$$\Delta V(n, x(n)) \leq -p(n)(W_3(|x(n)|) - \sigma).$$

Then,

(1) *if $p : Z^+ \to \mathbb{R}^+$ and $\sum_{m=0}^{\infty} p(m) = \infty$, then the solutions of equation (1) are uniformly bounded and ultimately bounded,*
(2) *if $p : Z^+ \to \mathbb{R}^+$ and for any $L > 0$, there exists an $L' \in Z^+ \backslash \{0\}$ such that $\sum_{m=n}^{n+L'} p(m) > L$ for $n \in Z^+$, then the solutions of equation (1) are uniformly bounded and uniformly ultimately bounded.*

The proof of Theorem 4 is similar to that of Theorem 3, so we omit it.

Now, we give a boundedness theorem similar to Corollary 3.

COROLLARY 5. *Let W_1, W_2, and W_3 be the same as in Theorem 1. If there exist a $V : Z^+ \times \mathbb{R}^k \to \mathbb{R}^+$ and a constant M such that*

(i) $W_1(|x(n)|) \leq V(n, x(n)) \leq W_2(|x(n)|)$ *for* $n \in Z^+$,
(ii) $\Delta V(n, x(n)) \leq p(n)G(V(n, x(n)), \sup_{m \in \{-\ell, \dots, 0\}} V(n + m, x(n + m))) + p(n)M$, *where* $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is continuous and* $G(V, V) = -W_3(V)$ *for* $V > 0$,

then

(1) *if $p : Z^+ \to \mathbb{R}^+$ and $\sum_{m=0}^{\infty} p(m) = \infty$, then the solutions of equation (1) are uniformly bounded and ultimately bounded,*
(2) *if $p : Z^+ \to \mathbb{R}^+$ and for any $L > 0$, there exist an $L' \in Z^+ \backslash \{0\}$ such that $\sum_{m=n}^{n+L'} p(m) \geq L$ for $n \in Z^+$, then the solutions of equation (1) are uniformly bounded and uniformly ultimately bounded.*

REMARK 5. The Δ in Corollary 5 can be the forward difference operator.

4. AN EXAMPLE

Consider the difference equation with finite delay of the form

$$x(n+1) = a(n)x(n) + b(n)x(n-\ell) + f(n), \qquad \text{for } n \in Z^+, \tag{4}$$

where $a, b, f : Z^+ \to \mathbb{R}$, $|a(n)| < 1$ and $\ell \in Z^+ \setminus \{0\}$. Suppose that

(i) there exists a $\mu < 1$ such that $|b(n)| \leq \mu(1 - |a(n)|)$,
(ii) $\sum_{m=0}^{\infty}(1 - |a(m)|) = \infty$,
(iii) for any $L > 0$, there exists an $L' \in Z^+ \setminus \{0\}$ such that

$$\sum_{m=n}^{n+L'} (1 - |a(m)|) > L, \qquad \text{for } n \in Z^+,$$

(iv) $|f(n)| \leq M(1 - |a(n)|)$ for $n \in Z^+$, where $M = \text{const} > 0$.

Then, we have the following propositions.

PROPOSITION 1. *If $f(n) \equiv 0$, then Conditions (i) and (ii) imply that the zero solution of equation (4) is uniformly stable and asymptotically stable; Conditions (i) and (iii) imply that the zero solution of equation (4) is uniformly asymptotically stable.*

PROOF. Let $V(x(n)) = |x(n)|$. Then,

$$\begin{aligned}\Delta V(x(n)) &= |x(n+1)| - |x(n)| \le -(1-|a(n)|)|x(n)| + |b(n)|\,|x(n-\ell)| \\ &\le -(1-|a(n)|)|x(n)| + \mu(1-|a(n)|)\|x_n\| \\ &\le (1-|a(n)|)G\left(V(x(n)), \sup_{m\in\{-\ell,\dots,0\}} V(x(n+m))\right),\end{aligned}$$

where $G(s,t) = -s + \mu t$. It is obvious that $G(s,t)$ is continuous for $s > 0$ and $t > 0$ and $G(V,V) = -(1-\mu)V$. This completes the proof by Corollary 3. ∎

Similarly, we can give the following proposition by Corollary 5.

PROPOSITION 2. *Conditions (i), (ii), and (iv) imply that the solutions of equation (4) are uniformly bounded and ultimately bounded; Conditions (i), (iii), and (iv) imply that the solutions of equation (4) are uniformly bounded and uniformly ultimately bounded.*

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